WEYL MODULES FOR THE TWISTED LOOP ALGEBRAS

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Abstract. The notion of a Weyl module, previously defined for the untwisted affine algebras, is extended here to the twisted affine algebras. We describe an identification of the Weyl modules for the twisted affine algebras with suitably chosen Weyl modules for the untwisted affine algebras. This identification allows us to use known results in the untwisted case to compute the dimensions and characters of the Weyl modules for the twisted algebras.

1. Introduction

The notion of Weyl modules for the untwisted affine Lie algebras was introduced in [6] and was motivated by an attempt to understand the category of finite dimensional representations of the untwisted quantum affine algebra. Specifically, the Weyl modules were conjectured to be the $q = 1$ limit of certain irreducible representations of the quantum affine algebras. It was proved that the conjecture was true for $\mathfrak{sl}_2$ and that this conjecture would follow if the dimensions of the Weyl modules were known. H. Nakajima has pointed out recently that the dimension formula follows by using results of [2] and [12].

Another approach to proving the dimension formula for the Weyl modules can be found in [4] for $\mathfrak{s}l_n$ and in [10] for the general simply laced case. These papers also make the connection between Weyl modules and the Demazure modules for affine Lie algebras and also with the fusion product defined by [7]. The approach in these papers is rather simple and show that one can study the Weyl modules from a purely classical viewpoint. Other points of interest and generalizations of these can be found in [8].

We now turn our attention to the case of the twisted affine algebras. None of the quantum machinery is available and in fact there are rather few results on the category of finite dimensional representations of the twisted quantum affine algebras [1], [5]. These results do show however that one can make a similar conjecture; i.e that one can define a notion of the Weyl module for the twisted affine Lie algebras such that they are the specializations of irreducible modules in the quantum case. To do this, one requires the Weyl modules to be universal in a suitable sense. One of the difficulties is in the case of the algebras of type $A_{2n}^{(2)}$, which are not built up entirely of algebras isomorphic to $A_1^{(1)}$; and indeed one needs to understand $A_2^{(2)}$ on its own. Thus, we use results of [9], [13] to arrive at the correct definition of the Weyl modules.

The next question clearly is to determine the dimensions of the Weyl modules and also their decomposition as modules for the underlying finite–dimensional simple Lie algebra. In the untwisted case these questions can be answered either by using the fusion product of [7] or the fact that the modules are specializations of modules for the quantum affine algebra. Both

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these techniques are unavailable to us in the twisted case, as far as we know the notion of fusion product does not admit a generalization to the twisted algebras. We get around these difficulties by identifying the Weyl modules for the twisted algebras $X_n^{(m)}$, $m>1$ with suitably chosen Weyl modules for the untwisted algebra $X_n^{(1)}$. We then use all the known results in the untwisted case to complete our analysis of the twisted algebras. In conclusion, we note that some of the methods we use in this paper give simpler proofs of some of the results in [6].

2. The untwisted loop algebras and the modules $W(\boldsymbol{\pi})$.

2.1. Throughout the paper $\mathbf{C}$ (resp. $\mathbf{C}^\times$) denotes the set of complex (resp. non–zero complex) numbers, and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$) the set of integers (resp. non–negative) integers. Given a Lie algebra $\mathfrak{a}$ we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $L(\mathfrak{a})$ denotes the loop algebra of $\mathfrak{a}$. Specifically, we have

$$L(\mathfrak{a})=\mathfrak{a}\otimes\mathbf{C}[t,t^{-1}],$$

with commutator given by

$$[x\otimes t^r,y\otimes t^s]=[x,y]\otimes t^{r+s}$$

for $x,y\in\mathfrak{a}$, $r,s\in\mathbf{Z}$. We identify $\mathfrak{a}$ with the subalgebra $\mathfrak{a}\otimes 1$ of $L(\mathfrak{a})$. Given $a\in\mathbf{C}^\times$, we let $\tau_a:L(\mathfrak{a})\to L(\mathfrak{a})$ be the automorphism defined by extending $\tau_a(x\otimes t^k)=a^k(x\otimes t^k)$ for all $x\in\mathfrak{g}$, $k\in\mathbf{Z}$.

Given $\ell,N\in\mathbf{Z}_+$ and $\mathbf{a}=(a_1,\cdots,a_\ell)\in(\mathbf{C}^\times)^\ell$ let $\mathfrak{a}_{\mathbf{a},N}$ be the quotient of $L(\mathfrak{a})$ by the ideal $\mathfrak{a}\otimes\prod_{k=1}^\ell(t-a_k)^N\mathbf{C}[t,t^{-1}]$.

Lemma. *Let $\mathbf{a}=(a_1,\cdots,a_\ell)\in(\mathbf{C}^\times)^\ell$ be such that $\mathbf{a}$ has distinct coordinates. For all $N\in\mathbf{Z}_+$, we have*

$$\mathfrak{a}_{\mathbf{a},N}\cong\oplus_{r=1}^N\mathfrak{a}_{a_r,N}.$$

Proof. Since $a_r\neq a_s$ if $1\le r\neq s\le\ell$, it is standard that

$$\mathbf{C}[t,t^{-1}]/\prod_{r=1}^{\ell}(t-a_r)^N\mathbf{C}[t,t^{-1}]\cong\oplus_{r=1}^\ell\mathbf{C}[t,t^{-1}]/(t-a_r)^N\mathbf{C}[t,t^{-1}]$$

and the lemma now follows trivially. □

2.2. The simple Lie algebras and their representations. Let $\mathfrak{g}$ be any finite-dimensional complex simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $W_\mathfrak{g}$ the corresponding Weyl group. Let $R_\mathfrak{g}$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $I_\mathfrak{g}$ an index set for a set of simple roots (and hence also for the fundamental weights), $R_\mathfrak{g}^+$ the set of positive roots, $Q_\mathfrak{g}^+$ (resp. $P_\mathfrak{g}^+$) the $\mathbf{Z}_+$ span of the simple roots (resp. fundamental weights) and $\theta_\mathfrak{g}$ be the highest root in $R_\mathfrak{g}^+$. Given $\alpha\in R_\mathfrak{g}$ let $\mathfrak{g}_\alpha$ be the corresponding root space, we have

$$\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+,\quad\mathfrak{n}^\pm=\bigoplus_{\alpha\in R^+}\mathfrak{g}_{\pm\alpha}.$$

Fix a Chevalley basis $x_\alpha^\pm,h_\alpha$, $\alpha\in R^+$ for $\mathfrak{g}$ and set

$$x_{\alpha_i}^\pm=x_i^\pm,\quad h_{\alpha_i}=h_i,\quad i\in I.$$

In particular for $i \in I$,
$$[x_i^+, x_i^-] = h_i, \quad [h_i, x_i^{\pm}] = \pm 2x_i^{\pm}.$$

Given a finite–dimensional representation of $\mathfrak{g}$ on a complex vector space V, we can write
$$V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v \ \ \forall\ h \in \mathfrak{h}\}.$$
Set $\mathrm{wt}(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$. It is well–known that
$$V_\mu \neq 0 \quad \Longrightarrow \mu \in P \ \text{ and } \ w\mu \in \mathrm{wt}(V) \ \ \forall \ \ w \in W,$$
and that V is isomorphic to a direct sum of irreducible representations. The set of isomorphism classes of irreducible finite–dimensional $\mathfrak{g}$–modules is in bijective correspondence with P^+ and for any $\lambda \in P^+$ let $V_{\mathfrak{g}}(\lambda)$ be an element of the corresponding isomorphism class. Then $V_{\mathfrak{g}}(\lambda)$ is generated by an element v_λ satisfying the relations:
$$\mathfrak{n}^+.v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1}v_\lambda = 0. \tag{2.1}$$

2.3. Identities in $\mathbf{U}(L(\mathfrak{g})$. For $i \in I$ it is easy to see that the elements $\{x_i^{\pm} \otimes t^k, h_i \otimes t^k : k \in \mathbf{Z}_+\}$ span a subalgebra of $L(\mathfrak{g})$ which is isomorphic to $L(\mathfrak{sl}_2)$. We shall need the following formal power series in u with coefficients in $\mathbf{U}(L(\mathfrak{g}))$. For $i \in I$, set
$$\mathbf{p}_i^{\pm}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_i \otimes t^k}{k} u^k\right),$$
$$\mathbf{x}_i^{\pm}(u) = \sum_{k=0}^{\infty} (x_i^{\pm} \otimes t^k) u^{k+1}, \quad \widetilde{\mathbf{x}}_i^{\pm}(u) = \sum_{k=-\infty}^{\infty} (x_i^{\pm} \otimes t^k) u^{k+1}$$

Given a power series $\mathbf{f}$ in u with coefficients in an algebra A, let $(\mathbf{f})_m$ be the coefficient of u^m ($m \in \mathbf{Z}$). The following result was proved in [11, Lemma 7.5], (see [6, Lemma 1.3] for the formulation in this notation).

Lemma. *Let $r \in \mathbf{Z}_+$.*
$$(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}_i^+(u)\right)_{r+1} \mod \mathbf{U}(L(\mathfrak{g}))\widetilde{\mathbf{x}}_i^{+}(u).$$
□

2.4. The monoid $\mathcal{P}^+$. Let $\mathcal{P}^+$ be the monoid of I–tuples of polynomials $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n)$ in an indeterminate u with constant term one, with multiplication being defined component wise. For $i \in I$ and $a \in \mathbf{C}^\times$, set
$$\boldsymbol{\pi}_{i,a} = ((1 - au)^{\delta_{ij}} : j \in I) \in \mathcal{P}^+, \tag{2.2}$$
and for $\lambda \in P^+$, set
$$\boldsymbol{\pi}_{\lambda,a} = \prod_{i \in I} (\boldsymbol{\pi}_{i,a})^{\lambda(h_i)}, \quad \lambda \neq 0.$$
Clearly any $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ can be written uniquely as a product
$$\boldsymbol{\pi}^+ = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i, a_i},$$

for some $\lambda_1, \cdots, \lambda_\ell \in P^+$ and distinct elements $a_1, \cdots, a_\ell \in \mathbf{C}^\times$ and in this case we set $\boldsymbol{\pi}^- = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i,\, a_i^{-1}}$. Define a map $\mathcal{P}^+ \to P^+$ by $\boldsymbol{\pi} \to \lambda_{\boldsymbol{\pi}} = \sum_{i\in I} \deg(\pi_i)\omega_i$.

2.5. The modules $W(\boldsymbol{\pi})$, $V(\boldsymbol{\pi})$. Given $\boldsymbol{\pi} = (\pi_i)_{i\in I} \in \mathcal{P}^+$, let $W(\boldsymbol{\pi})$ be the $L(\mathfrak{g})$–module generated by an element $w_{\boldsymbol{\pi}}$ with relations:

$$L(\mathfrak{n}^+)w_{\boldsymbol{\pi}} = 0, \quad hw_{\boldsymbol{\pi}} = \lambda_{\boldsymbol{\pi}}(h)w_{\boldsymbol{\pi}}, \quad (x_i^-)^{\lambda_{\boldsymbol{\pi}}(h_i)+1}w_{\boldsymbol{\pi}} = 0,$$
$$\left(\mathbf{p}_i^{\pm}(u) - \pi_i^{\pm}(u)\right) w_{\boldsymbol{\pi}} = 0,$$

where $\lambda_{\boldsymbol{\pi}} = \sum_{i\in I}(\deg \pi_i)\omega_i$, $\boldsymbol{\pi}^+ = \boldsymbol{\pi}$, $i \in I$ and $h \in \mathfrak{h}$. It is not hard to see that if we write $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_\ell, a_\ell}$ where $a_1, \cdots, a_\ell$ are all distinct, then for $i \in I$

$$\left(\mathbf{p}_i^{\pm}(u) - \pi_i^{\pm}(u)\right) w_{\boldsymbol{\pi}} = 0 \iff (h_i \otimes t^r)w_{\boldsymbol{\pi}} = \left(\sum_{j=1}^{\ell} \lambda_j(h_i)a_j^r\right) w_{\boldsymbol{\pi}}.$$

Let $b \in \mathbf{C}^\times$ and let $\tau_b W(\boldsymbol{\pi})$ be the $L(\mathfrak{g})$–module obtained by pulling back $W(\boldsymbol{\pi})$ through the automorphism τ_b of $L(\mathfrak{g})$. The next result is standard.

Lemma. (i) *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. Then $W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{n}^-))w_{\boldsymbol{\pi}}$, and hence we have,*

$$\mathrm{wt}(W(\boldsymbol{\pi}) \subset \lambda_{\boldsymbol{\pi}} - Q^+, \quad \dim W(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}} = 1.$$

In particular, the module $W(\boldsymbol{\pi})$ has a unique irreducible quotient $V(\boldsymbol{\pi})$.

(ii) *For $b \in \mathbf{C}^\times$, we have $\tau_b W(\boldsymbol{\pi}) \cong W(\boldsymbol{\pi}_b)$, where $\boldsymbol{\pi} = (\pi_i(u))_{i\in I}$ and $\boldsymbol{\pi}_b = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have*

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{\lambda,ab}).$$

2.6. The modules $W(\boldsymbol{\pi})$ were initially defined and studied in [6] and a formula was conjectured for their dimension. Parts (i) and (ii) of the next theorem were proved in [6]. Part (iii) was proved in [6] in the case of $\mathfrak{sl}_2$, for $\mathfrak{sl}_\mathfrak{n}$ it was proved in [4] and for the general simply laced case in [10]. Part (iii) can be deduced for the general case by using results of [2],[12],[14] for quantum affine algebras.

Theorem 1. (i) Given $\boldsymbol{\pi} = (\pi_i)_{i\in I}$ with unique decomposition $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_\ell, a_\ell}$, we have an isomorphism of $L(\mathfrak{g})$–modules

$$W(\boldsymbol{\pi}) \cong \otimes_{k=1}^{\ell} W(\boldsymbol{\pi}_{\lambda_k, a_k}).$$

(ii) Let V be any finite–dimensional $L(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L(\mathfrak{n}^+)v = 0, \quad L(\mathfrak{h})v = \mathbf{C}v.$$

Then there exists $\boldsymbol{\pi} \in \mathcal{P}^+$ such that the assignment $w_{\boldsymbol{\pi}} \to v$ extends to a surjective homomorphism $W(\boldsymbol{\pi}) \to V$ of $L(\mathfrak{g})$–modules.

(iii) Let $\lambda \in P^+$ and $a \in \mathbf{C}^\times$. Suppose that $\lambda = \sum_{i\in I} m_i\omega_i$. Then

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} \bigotimes_{i\in I} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i}.$$

$\square$

2.7. Annihilating ideals for $W(\boldsymbol{\pi})$. The next proposition is implicit in [6] but since it plays a big role in this paper we make it explicit and give a proof.

Proposition. *Let $\boldsymbol{\pi} = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r, a_r} \in \mathcal{P}^+$. There exists an integer $N = N(\boldsymbol{\pi})$ such that*

$$\left(\mathfrak{g} \otimes \prod_{r=1}^{\ell}(t-a_r)^N \mathbf{C}[t,t^{-1}]\right) W(\boldsymbol{\pi}) = 0.$$

Proof. We begin by proving that for all $i \in I$

$$x_i^- \otimes \prod_{r=1}^{\ell}(t-a_r)^{\lambda_r(h_i)} w_{\boldsymbol{\pi}} = 0. \tag{2.3}$$

Set $N_i = \lambda_{\boldsymbol{\pi}}(h_i)$. Using the defining relations of $W(\boldsymbol{\pi})$ and Lemma 2.3,

$$0 = (x_i^+ \otimes t)^{N_i}(x_i^- \otimes 1)^{N_i+1} w_{\boldsymbol{\pi}} = (-1)^{N_i}\left(\mathbf{x}_i^-(u)\mathbf{p}_i^+(u)\right)_{N_i} w_{\boldsymbol{\pi}}.$$

We also have

$$\mathbf{p}_i(u).w_{\boldsymbol{\pi}} = \prod_{r=1}^{\ell}(1-a_r u)^{\lambda_r(h_i)}.w_{\boldsymbol{\pi}} \equiv \left(\sum_{j=0}^{N_i} p_{i,j} u^j\right).w_{\boldsymbol{\pi}}.$$

Combining these we get

$$\left(\mathbf{x}_i^-(u)\mathbf{p}_i^+(u)\right)_{N_i} w_{\boldsymbol{\pi}} = \left(\sum_{j=0}^{N_i} x_i^- \otimes p_{i,N_i-j} t^j\right) w_{\boldsymbol{\pi}} = x_i^- \otimes \left(\sum_{j=0}^{N_i} t^j p_{i,N_i-j}\right) w_{\boldsymbol{\pi}} = 0.$$

But it is elementary to see that

$$\sum_{j=0}^{N_i} t^j p_{i,N_i-j} = \prod_{r=1}^{\ell}(t-a_r)^{\lambda_r(h_i)},$$

which proves (2.3). Since $\mathfrak{n}^-$ is generated by the elements x_i^-, $i \in I$, it is immediate that there exists $N \gg 0$ such that

$$\left(x_\theta^- \otimes \prod_{r=1}^{\ell}(t-a_r)^N\right) w_{\boldsymbol{\pi}} = 0. \tag{2.4}$$

Since $[\mathfrak{n}^-, x_\theta^-] = 0$ and $W(\boldsymbol{\pi}) \cong \mathbf{U}(L(\mathfrak{n}^-))w_{\boldsymbol{\pi}}$ as vector spaces, we get

$$\left(x_\theta^- \otimes \prod_{r=1}^{\ell}(t-a_r)^N\right) W(\boldsymbol{\pi}) = 0.$$

Since any element in $\mathfrak{g}$ is in the span of elements of the form $\{[x_{i_1}^+[x_{i_2}^+[\cdots[x_{i_k}^+, x_\theta^-], \cdots]] : i_1, \cdots, i_k \in I\}$, we now get

$$\left(\mathfrak{g} \otimes \prod_{r=1}^{\ell}(t-a_r)^N \mathbf{C}[t,t^-1]\right) W(\boldsymbol{\pi}) = 0.$$

$\square$

Corollary. *Given $\boldsymbol{\pi} \in \mathcal{P}^+$ with unique decomposition $\boldsymbol{\pi} = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r, a_r} \in \mathcal{P}^+$, there exists $N \in \mathbf{Z}_+$ such that the action of $L(\mathfrak{g})$ on $W(\boldsymbol{\pi})$ factors through to an action of $\mathfrak{g}_{\mathbf{a},N}$ on $W(\boldsymbol{\pi})$ and $W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{n}^-_{\mathbf{a},N}))w_{\boldsymbol{\pi}}$.*

3. The twisted algebras $L^\sigma(\mathfrak{g})$ and the modules $W(\boldsymbol{\pi}^\sigma)$

3.1. Assume from now on that $\mathfrak{g}$ is simply–laced and that $\sigma : \mathfrak{g} \to \mathfrak{g}$ is a non–trivial diagram automorphism of $\mathfrak{g}$ of order m. In particular σ induces a permutation of I and R^+ and we have

$$\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}, \quad \sigma(\mathfrak{h}) = \mathfrak{h}, \ \sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\pm.$$

Let ζ be a primitive m^{th} root of unity, we have

$$\mathfrak{g} = \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon, \quad \mathfrak{g}_\epsilon = \{x \in \mathfrak{g} : \sigma(x) = \zeta^\epsilon x\}.$$

Given any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is preserved by σ, set $\mathfrak{a}_\epsilon = \mathfrak{g}_\epsilon \cap \mathfrak{a}$. It is known that $\mathfrak{g}_0$ is a simple Lie algebra, $\mathfrak{h}_0$ is a Cartan subalgebra and that $\mathfrak{g}_\epsilon$ is an irreducible representation of $\mathfrak{g}_0$ for all $0 \ leq \epsilon \leq m-1$. Moreover,

$$\mathfrak{n}^\pm \cap \mathfrak{g}_0 = \mathfrak{n}_0^\pm = \bigoplus_{\alpha \in R^+_{\mathfrak{g}_0}} (\mathfrak{g}_0)_{\pm\alpha}.$$

The following table describes the various possibilities for $\mathfrak{g}$, $\mathfrak{g}_0$ and the structure of $\mathfrak{g}_k$ as a $\mathfrak{g}_0$–module, here θ_0^s is the highest short root of $\mathfrak{g}_0$ and $B_1 = A_1$.

m	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_k$
2	$A_{2n},$	B_n	$V_{\mathfrak{g}_0}(2\theta_0^s)$
2	$A_{2n-1}, \ n \geq 2$	C_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	$D_{n+1}, \ n \geq 3$	B_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	E_6	F_4	$V_{\mathfrak{g}_0}(\theta_0^s)$
3	D_4	G_2	$V_{\mathfrak{g}_0}(\theta_0^s)$

From now we set $R_{\mathfrak{g}} = R^+$, $R_{\mathfrak{g}_0} = R_0$, the sets I, P^+ etc. are defined similarly. The set of σ– orbits of I has the same cardinality as I_0 and we identify I_0 with a subset of I. In the case when $\mathfrak{g}$ is of type A_{2n} we assume that $n \in I_0$ corresponds to the unique short simple root of $\mathfrak{g}_0$. We shall also fix ζ a primitive m^{th} root of unity.

Suppose that $\{y_i : i \in I\}$ is one of the sets $\{h_i : i \in I\}$, $\{x_i^+ : i \in I\}$ or $\{x_i^- : i \in I\}$ and assume that $m = 2$ and that $i \neq n$ if $\mathfrak{g}$ is of type A_{2n}. Define subsets $\{y_{i,\epsilon} : i \in I_0, \ 0 \leq \epsilon \leq 1\}$ of $\mathfrak{g}_\epsilon$ by

$$y_{i,0} = y_i \quad \text{if } i = \sigma(i), \qquad y_{i,0} = y_i + y_{\sigma(i)} \quad \text{if } i \neq \sigma(i),$$
$$y_{i,1} = y_i - y_{\sigma(i)} \quad \text{if } i \neq \sigma(i) \qquad y_{i,1} = 0 \quad \text{if } i = \sigma(i),$$

If $\mathfrak{g}$ is of type A_{2n}, then we set,

$$h_{n,0} = 2(h_n + h_{n+1}), \quad x^{\pm}_{n,0} = \sqrt{2}(x^{\pm}_n + x^{\pm}_{n+1}),$$
$$x^{\pm}_{n,1} = -\sqrt{2}(x^{\pm}_n - x^{\pm}_{n+1}), \quad h_{n,1} = h_n - h_{n+1},$$
$$y^{\pm}_{n,1} = \mp\frac{1}{4}\left[x^{\pm}_{n,0}, x^{\pm}_{n,1}\right].$$

Finally if $\mathfrak{g}$ is of type D_4 and $m = 3$, set,

$$y_{i,0} = y_i \quad \text{if } i = \sigma(i), \qquad y_{i,0} = \sum_{j=0}^{m-1} y_{\sigma^j(i)} \quad \text{if } i \neq \sigma(i),$$
$$y_{i,1} = y_{i,2} = 0 \quad \text{if } i = \sigma(i),$$
$$y_{i,1} = y_i + \zeta^2 y_{\sigma(i)} + \zeta y_{\sigma^2(i)}, \qquad y_{i,2} = y_i + \zeta y_{\sigma(i)} + \zeta^2 y_{\sigma^2(i)} \quad \text{if } i \neq \sigma(i),$$

In the rest of this paper in the case when $\mathfrak{g}$ is of type A_{2n}, we shall only be interested in elements $\lambda \in P_0^+$ such that $\lambda(h_{n,0}) \in 2\mathbf{Z}_+$ and we let P_σ^+ denote this subset of P_0^+. Moreover we regard $\lambda \in P_\sigma^+$ as an element of P^+ as follows:

$$\lambda(h_i) = \begin{cases} \lambda(h_{i,0}), \quad i \in I_0, \quad \textit{if } \mathfrak{g} \textit{ is not of type } A_{2n} \\ 0 \quad i \notin I_0, \\ (1 - \delta_{i,n}/2)\lambda(h_{i,0}), \ \textit{if } \mathfrak{g} \textit{ is of type } A_{2n}. \end{cases}$$

3.2. Let $\widetilde{\sigma} : L(\mathfrak{g}) \to L(\mathfrak{g})$ be the automorphism defined by extending,

$$\widetilde{\sigma}(x \otimes t^k) = \zeta^k \sigma(x) \otimes t^k,$$

for $x \in \mathfrak{g}$, $k \in \mathbb{Z}$. Then $\tilde{\sigma}$ is or order m and we let $L^\sigma(\mathfrak{g})$ be the subalgebra of fixed points of $\tilde{\sigma}$. Clearly,

$$L^\sigma(\mathfrak{g}) \cong \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\mathbf{C}[t^m, t^{-m}].$$

Lemma. *Let $i \in I_0$ and assume that $i \neq n$ if $\mathfrak{g}$ is of type A_{2n}. The subalgebra of $L^\sigma(\mathfrak{g})$) spanned by the elements $\{x^{\pm}_{i,\epsilon} \otimes t^{mk-\epsilon}, h_{i,\epsilon} \otimes t^{mk-\epsilon} : k \in \mathbf{Z},\ 0 \leq \epsilon \leq m-1\}$ is canonically isomorphic to $L(\mathfrak{sl}_2)$. If $\mathfrak{g}$ is of type A_{2n} the subalgebra of $L^\sigma(\mathfrak{g})$) spanned by the elements $\{x^{\pm}_{n,\epsilon} \otimes t^{2k+\epsilon}, h_{n,\epsilon} \otimes t^{2k+\epsilon}, \mp\frac{1}{4}[x^{\pm}_{n,0}, x^{\pm}_{n,1}] \otimes t^{2k+1} : k \in \mathbf{Z},\ 0 \leq \epsilon \leq m-1\}$ is canonically isomorphic to $L^\sigma(\mathfrak{sl}_3)$.* □

3.3. Identities in $\mathbf{U}(L^\sigma(\mathfrak{g}))$. Suppose that either $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)^+_s$ or that $\mathfrak{g}$ is of type A_{2n} and $i \neq n$. Define power series with coefficients in $\mathbf{U}(L(^\sigma(\mathfrak{g}))$ by,

$$\mathbf{p}^{\pm}_{i,\sigma}(u) = \exp\left(-\sum_{k=1}^{\infty}\sum_{\epsilon=0}^{m-1} \frac{h_{i,\epsilon} \otimes t^{mk-\epsilon}}{mk-\epsilon} u^{mk-\epsilon}\right),$$

$$\mathbf{x}^-_i(u) = \sum_{k=0}^{\infty}\sum_{\epsilon=0}^{m-1} (x^-_{i,m-\epsilon} \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}, \qquad \tilde{\mathbf{x}}^+_i(u) = \sum_{k=-\infty}^{\infty}\sum_{\epsilon=0}^{m-1} (x^+_{i,m-\epsilon} \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}.$$

If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_l^+$, then we set

$$\mathbf{p}_{i,\sigma}^{\pm}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{i,0}\otimes t^{mk}}{k} u^k\right),$$

$$\mathbf{x}_i^-(u) = \sum_{k=0}^{\infty} (x_{i,0}^- \otimes t^{mk} u^{k+1}) \qquad \tilde{\mathbf{x}}_i^+(u) = \sum_{k=0}^{\infty} (x_{i,0}^+ \otimes t^{mk}) u^{k+1}.$$

Finally, if $\mathfrak{g}$ is of type A_{2n} and $i = n$ we have,

$$\mathbf{p}_{n,\sigma}^{\pm}(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{n,0}/2 \otimes t^{2k}}{2k} u^{2k} + \sum_{k=1}^{\infty} \frac{h_{n,1}\otimes t^{2k-1}}{2k-1} u^{2k-1}\right),$$

$$\mathbf{x}_n^-(u) = \sum_{k=0}^{\infty}\sum_{\epsilon=0}^{m-1} (x_{n,\epsilon}^- \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}, \qquad \tilde{\mathbf{x}}_n^+(u) = \sum_{k=-\infty}^{\infty}\sum_{\epsilon=0}^{m-1} (x_{n,\epsilon}^+ \otimes t^{mk+\epsilon}) u^{mk+\epsilon+1}.$$

Lemma. *Let $r \in \mathbf{Z}_+$.*

(i) *If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_s^+$ or $\mathfrak{g}$ is of type A_{2n} and $\alpha_i \in (R_0)_l^+$, we have*

$$(x_{i,1}^+ \otimes t)^{(r)} (x_{i,0}^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}_{i,\sigma}^+(u)\right)_{r+1} \quad \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_i{}^+(u).$$

(ii) *If $\mathfrak{g}$ is not of type A_{2n} and $\alpha_i \in (R_0)_l^+$,*

$$(x_{i,0}^+ \otimes t^2)^{(r)} (x_{i,0}^- \otimes 1)^{(r+1)} = (-1)^r \left(\mathbf{x}_i^-(u)\mathbf{p}_{i,\sigma}^+(u)\right)_{r+1} \quad \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_i{}^+(u).$$

(iii) *If $\mathfrak{g}$ is of type A_{2n}, we have*

(a) $(x_{n,0}^+ \otimes 1)^{(2r-1)} (y_{n,1}^- \otimes t)^{(r)} = -\left(\mathbf{x}_n^-(u)\mathbf{p}_{n,\sigma}^+(u)\right)_r \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_n^+(u),$

(b) $(x_{n,0}^+ \otimes 1)^{(2r)} (y_{n,1}^- \otimes t)^{(r)} = -\left(\mathbf{p}_{n,\sigma}^+(u)\right)_r \mod \mathbf{U}(L^\sigma(\mathfrak{g}))\widetilde{\mathbf{x}}_n^+(u),$
where

$$y_{n,1}^- = \frac{1}{4}\left[x_{n,0}^-, x_{n,1}^-\right].$$

Proof. Parts (i) and (ii) are immediate consequences of Lemma 2.3 and Lemma 3.2. Part (iii) is deduced from [13], [9, Lemma 5.36], exactly as (i) and (ii) were deduced from Garland in [6]. □

3.4. The monoid $\mathcal{P}_\sigma^+$. Let $(\ ,\ )$ be the form on $\mathfrak{h}_0^*$ induced by the Killing form of $\mathfrak{g}_0$ normalized so that $(\theta_0, \theta_0) = 2$. For $i \in I_0$ and $a \in \mathbf{C}^\times$, $\lambda \in P_0^+$ and $\mathfrak{g}$ not of type A_{2n} let

$$\boldsymbol{\pi}_{i,a}^\sigma = ((1 - a^{(\alpha_i,\alpha_i)} u)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}_{\lambda,a}^\sigma = \prod_{i\in I_0} \left(\boldsymbol{\pi}_{i,a}^\sigma\right)^{\lambda(h_i)},$$

while if $\mathfrak{g}$ is of type A_{2n} we set for $i \in I_0$, $a \in \mathbf{C}^\times$, $\lambda \in P_\sigma^+$,

$$\boldsymbol{\pi}_{i,a}^\sigma = ((1 - au)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}_{\lambda,a}^\sigma = \prod_{i\in I_0} \left(\boldsymbol{\pi}_{i,a}^\sigma\right)^{(1-\frac{1}{2}\delta_{i,n})\lambda(h_i)}.$$

Let $\mathcal{P}_\sigma^+$ be the monoid generated by the elements $\boldsymbol{\pi}_{\lambda,a}^\sigma$. Define a map $\mathcal{P}_\sigma^+ \to P_\sigma^+$ by

$$\lambda_{\boldsymbol{\pi}^\sigma} = \sum_{i\in I_0} (\deg \pi_i)\omega_i,$$

if $\mathfrak{g}$ is not of type A_{2n} and

$$\lambda_{\boldsymbol{\pi}^\sigma} = \sum_{i\in I_0} (1+\delta_{i,n})(\deg \pi_i)\omega_i,$$

if $\mathfrak{g}$ is of type A_{2n}. It is clear that any $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ can be written (non–uniquely) as product

$$\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}^\sigma,$$

where $\mathbf{a} = (a_1, \cdots, a_\ell)$ and $\mathbf{a}^m$ have distinct coordinates. We call any such expression a standard decomposition of $\boldsymbol{\pi}^\sigma$.

3.5. The set $\mathbf{i}(\boldsymbol{\pi}^\sigma)$. Given $\lambda = \sum_{i\in I} m_i\omega_i \in P^+$ and $0 \le \epsilon \le m-1$, define elements $\lambda(\epsilon) \in P_\sigma^+$ by,

$$\lambda(0) = \sum_{i\in I_0} m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i} m_{\sigma(i)}\omega_i, \quad \text{if } m=2 \text{ and } \mathfrak{g} \text{ not of type } A_{2n}$$

$$\lambda(0) = \sum_{i\in I_0} (1+\delta_{i,n})m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i} (1+\delta_{\sigma(i),n})m_{\sigma(i)}\omega_i, \quad \text{if } m=2 \text{ and } \mathfrak{g} \text{ of type } A_{2n}$$

$$\lambda(0) = m_1\omega_1 + m_2\omega_2, \quad \lambda(1) = m_3\omega_1, \quad \lambda(2) = m_4\omega_1, \quad \text{if } m=3.$$

Define a map $\mathbf{r} : \mathcal{P}^+ \to \mathcal{P}_\sigma^+$ as follows. Given $\boldsymbol{\pi} \in \mathcal{P}^+$ write

$$\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k,a_k}, \quad a_k \neq a_p, \quad 1 \le k \neq p \le \ell,$$

and set

$$\mathbf{r}(\boldsymbol{\pi}) = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}_{\lambda_k(\epsilon),\zeta^\epsilon a_k}^\sigma.$$

Note that $\mathbf{r}$ is well defined since the choice of (λ_k, a_k) is unique and set

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \{\boldsymbol{\pi} \in \mathcal{P}^+ : \mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma\}.$$

We now give an explicit description of the set $\mathbf{i}(\boldsymbol{\pi}^\sigma)$. Recall that given $\lambda \in P_\sigma^+$, we also regard $\lambda \in P^+$ as in Section 3.1. In addition, define $\sigma(\omega_i) = \omega_{\sigma(i)}$ for $i \in I$.

Lemma. (i) *Let $i \in I_0$ and $a \in \mathbf{C}^\times$. We have,*

$$\mathbf{i}(\boldsymbol{\pi}_{\omega_i,a}^\sigma) = \{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{m-r}a} \,|\, 0 \le r < m\},$$

and for A_{2n}^2 and $i = n$,

$$\mathbf{i}(\boldsymbol{\pi}_{2\omega_n,a}^\sigma) = \{\boldsymbol{\pi}_{\omega_n,a}\,, \boldsymbol{\pi}_{\omega_{n+1},-a}\}$$

(ii) *Let* $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}(\boldsymbol{\pi}^\sigma_{\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ *be a decomposition of* $\boldsymbol{\pi}^\sigma$ *into linear factors for* $\mathfrak{g}$ *not of type* A_{2n}*. Then*

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{m-r+\epsilon}a_k}\,|\,0\le r<m\}^{m_{k,\epsilon,i}}$$

where the product of the sets is understood to be the set of products of elements of the sets.

In the case of $A_{2n}^{(2)}$*, let* $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{1}\prod_{i\in I_0}(\boldsymbol{\pi}^\sigma_{(1+\delta_{i,n})\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ *be a decomposition of* $\boldsymbol{\pi}^\sigma$ *into linear factors. Then*

$$\mathbf{i}(\boldsymbol{\pi}^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{2}\prod_{i\in I_0}\{\boldsymbol{\pi}_{\sigma^r(\omega_i),\zeta^{2-r+\epsilon}a_k}\,|\,0\le r<2\}^{m_{k,\epsilon,i}}$$

(iii) *In particular, we have*

$$\prod_{k=1}^{\ell}\boldsymbol{\pi}_{\mu_k,a_k} = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\boldsymbol{\pi}^{m_{k,\epsilon,i}}_{\sigma^\epsilon(\omega_i),a_k}\in\mathbf{i}(\boldsymbol{\pi}^\sigma),$$

where $\mu_k = \sum_{\epsilon=0}^{m-1}\sum_{i\in I_0} m_{k,\epsilon,i}\sigma^\epsilon(\omega_i)$ *and* $a_i^m \neq a_j^m$*.*

Proof. The first statement is trivially checked, noting that if i is a fixed point of σ, then $\boldsymbol{\pi}^\sigma_{\omega_i,a} = \boldsymbol{\pi}^\sigma_{\omega_i,\zeta^r a}$ for $0\le r<m$. The other statements follow immediately from the first one. $\square$

From here on we shall assume that, unless otherwise noted, the element $\boldsymbol{\pi}\in\mathbf{i}(\boldsymbol{\pi}^\sigma)$ chosen is of the form given in (iii) of the lemma.

3.6. The modules $W(\boldsymbol{\pi}^\sigma)$**,** $V(\boldsymbol{\pi}^\sigma)$**.** Given $\boldsymbol{\pi}^\sigma = (\pi_{i,\sigma})_{i\in I_0}\in\mathcal{P}^+_\sigma$, the Weyl module $W(\boldsymbol{\pi}^\sigma)$ is the $\mathbf{U}(L^\sigma(\mathfrak{g}))$-module generated by an element $w_{\boldsymbol{\pi}^\sigma}$ with relations:

$$L^\sigma(\mathfrak{n}^+)w_{\boldsymbol{\pi}^\sigma}=0,\quad hw_{\boldsymbol{\pi}} = \lambda_{\boldsymbol{\pi}}(h)w_{\boldsymbol{\pi}^\sigma},\quad (x^-_{i,0})^{\lambda_{\boldsymbol{\pi}}(h_i)+1}w_{\boldsymbol{\pi}^\sigma}=0,$$

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w_{\boldsymbol{\pi}^\sigma}=0,$$

for all $i\in I_0$ and $h\in\mathfrak{h}_0$. If $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\boldsymbol{\pi}^\sigma_{\lambda_k,a_k}\in\mathcal{P}^+_\sigma$, it is not hard to see that for $i\in I_0$, we have if $\mathfrak{g}$ not of type A_{2n},

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w_{\boldsymbol{\pi}^\sigma}=0 \iff (h_{i,\epsilon}\otimes t^{mk-\epsilon})w_{\boldsymbol{\pi}^\sigma} = \sum_{j=1}^{\ell}\lambda_j(h_{i,0})a_j^{mk-\epsilon}w_{\boldsymbol{\pi}^\sigma}, \tag{3.1}$$

and for $\mathfrak{g}$ of type A_{2n},

$$\left(\mathbf{p}^\pm_{i,\sigma}(u)-\pi^\pm_{i,\sigma}(u)\right)w^\sigma_{\lambda,a}=0 \iff (h_{i,\epsilon}\otimes t^{mk-\epsilon})w_{\boldsymbol{\pi}^\sigma} = \sum_{j=1}^{\ell}(1-\frac{1}{2}\delta_{i,n})\lambda_j(h_{i,\epsilon})a_j^{mk-\epsilon}w_{\boldsymbol{\pi}^\sigma}. \tag{3.2}$$

3.7. For $b \in \mathbf{C}^\times$ we have $\tau_b(L^\sigma(\mathfrak{g})) \subset L^\sigma(\mathfrak{g})$ and we let $\tau_b W(\boldsymbol{\pi}^\sigma)$ be the $L^\sigma(\mathfrak{g})$–module obtained by pulling back $W(\boldsymbol{\pi}^\sigma)$ through τ_b. The next result is proved by standard methods.

Lemma. (i) *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$. Then $W(\boldsymbol{\pi}^\sigma) = \mathbf{U}(L^\sigma(\mathfrak{n}^-))w^\sigma_{\boldsymbol{\pi}}$, and hence we have,*

$$\mathrm{wt}(W(\boldsymbol{\pi}^\sigma) \subset \lambda_{\boldsymbol{\pi}^\sigma} - Q_0^+, \qquad \dim W(\mathbf{P}^\sigma)_{\lambda_{\boldsymbol{\pi}^\sigma}} = 1.$$

In particular, the module $W(\boldsymbol{\pi}^\sigma)$ has a unique irreducible quotient $V(\boldsymbol{\pi}^\sigma)$.

(ii) *For $b \in \mathbf{C}^\times$, we have $\tau_b W(\boldsymbol{\pi}^\sigma) \cong W(\boldsymbol{\pi}^\sigma_b)$, where $\boldsymbol{\pi}^\sigma = (\pi_i(u))_{i\in I}$ and $\boldsymbol{\pi}^\sigma_b = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have*

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}^\sigma_{\lambda,ba}).$$

□

3.8. The main theorem. In the rest of this paper we shall prove the following result.

Theorem 2. (i) Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$. For all $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$, we have

$$W(\boldsymbol{\pi}^\sigma) \cong_{L^\sigma(\mathfrak{g})} W(\boldsymbol{\pi}), \qquad V(\boldsymbol{\pi}^\sigma) \cong_{L^\sigma(\mathfrak{g})} V(\boldsymbol{\pi}).$$

(ii) Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ and assume that $\prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k} \in \mathcal{P}_\sigma^+$ is a standard decomposition of $\boldsymbol{\pi}$. As $L^\sigma(\mathfrak{g})$–modules, we have

$$W(\boldsymbol{\pi}^\sigma) \cong \bigotimes_{k=1}^{\ell} W(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}).$$

(iii) Suppose that $\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a} \in \mathcal{P}_\sigma^+$. Then

$$W(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \cong_{\mathfrak{g}_0} \bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}).$$

(iv) Let $\lambda = \sum_{i\in I_0} m_i\omega_i \in P_\sigma^+$ and $a \in \mathbf{C}^\times$. We have for $\mathfrak{g}$ not of type A_{2n}

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} \bigotimes_{i=1}^{n} W(\boldsymbol{\pi}^\sigma_{\omega,1})^{\otimes m_i}$$

and for $\mathfrak{g}$ of type A_{2n}

$$W(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}^\sigma_{2\omega_n,1})^{\otimes\frac{m_n}{2}} \otimes \bigotimes_{i=1}^{n-1} W(\boldsymbol{\pi}^\sigma_{\omega_i,1})^{\otimes m_i}.$$

(v) Let V be any finite–dimensional $L^\sigma(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L^\sigma(\mathfrak{n}^+)v = 0, \quad L^\sigma(\mathfrak{h})v = \mathbf{C}v.$$

Then there exists $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$ such that the assignment $w_{\boldsymbol{\pi}^\sigma} \to v$ extends to a surjective homomorphism $W(\boldsymbol{\pi}^\sigma) \to V$ of $L^\sigma(\mathfrak{g})$–modules.

□

4. Proof of Theorem 2

4.1. Annihilating ideals for $W(\boldsymbol{\pi}^\sigma)$.

Proposition. *Let $\boldsymbol{\pi}^\sigma = \prod_{r=1}^{\ell} \boldsymbol{\pi}^\sigma_{\lambda_r,a_r} \in \mathcal{P}^+_\sigma$. There exists an integer $N = N(\boldsymbol{\pi})$ such that*

$$\left(\bigoplus_{\epsilon=0}^{m-1}(\mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\prod_{r=1}^{\ell}(t^m - a_r^m)^N \mathbf{C}[t^m, t^{-m}])\right) W(\boldsymbol{\pi}^\sigma) = 0.$$

Proof. The subalgebra $L^m(\mathfrak{g}_0) = \mathfrak{g}_0 \otimes \mathbf{C}[t^m, t^{-m}]$ is canonically isomorphic to $L(\mathfrak{g}_0)$. It follows from the defining relations that

$$L^m(\mathfrak{n}_0^+) w_{\boldsymbol{\pi}^\sigma} = 0, \qquad (h_0 \otimes t^{mk}) w_{\boldsymbol{\pi}^\sigma} = \left(\sum_{r=1}^{\ell} \lambda_r(h_0) a_r^{mk}\right) w_{\boldsymbol{\pi}^\sigma},$$

and hence, $\mathbf{U}(L^m(\mathfrak{g}_0)) w_{\boldsymbol{\pi}^\sigma}$ is a quotient of the $L(\mathfrak{g}_0)$–module $W_{\mathfrak{g}_0}(\boldsymbol{\pi}_m)$ where

$$\boldsymbol{\pi}_m = \prod_{r=1}^{\ell} \boldsymbol{\pi}_{\lambda_r, a_r^m}.$$

It follows from (2.4) that

$$(x_{\theta_0}^- \otimes \prod_{r=1}^{m}(t^m - a_r^m)) w_{\boldsymbol{\pi}}^\sigma = 0, \tag{4.1}$$

for some $N \in \mathbf{Z}_+$, where $\theta_0 \in R_0^+$ is the highest root in R_0^+.

Assume first that $\mathfrak{g}$ is not of type A_{2n}, then

$$[x_{\theta_0}^-, L^\sigma(\mathfrak{n}^-)] = 0, \qquad [\mathfrak{h}_\epsilon, \mathfrak{g}_0] = \mathfrak{g}_\epsilon, \quad 0 \le \epsilon \le m-1. \tag{4.2}$$

The first equality in (4.2) gives $\left(x_{\theta_0}^- \otimes \prod_{r=1}^{\ell}(t^m - a_r^m)^N\right) W(\boldsymbol{\pi}^\sigma) = 0,$. One deduces now as in the untwisted case that

$$\left(\mathfrak{g}_0 \otimes \left(\prod_{r=1}^{\ell}(t^m - a_r^m)^N\right) \mathbf{C}[t^m, t^{-m}]\right) W(\boldsymbol{\pi}^\sigma) = 0.$$

Applying $\mathfrak{h}_\epsilon \otimes t^{m-\epsilon}$ to the preceding equation and using the second equality in (4.2) gives

$$\left(\mathfrak{g}_\epsilon \otimes t^{m-\epsilon}\left(\prod_{r=1}^{\ell}(t^m - a_r^m)\right) \mathbf{C}[t^m, t^{-m}]\right) W(\boldsymbol{\pi}^\sigma) = 0,$$

for all $0 \le \epsilon \le m-1$ and the result is proved.

Assume now that $\mathfrak{g}$ is of type A_{2n}. This time, we use the fact that

$$(x_{n,\epsilon}^- \otimes t^\epsilon \mathbf{C}[t^2, t^{-2}]) w_{\boldsymbol{\pi}} \in \mathbf{U}(L^\sigma(\mathfrak{h} \oplus \mathfrak{n}^+))(x_{\theta_0}^- \otimes \mathbf{C}[t^2, t^{-2}]) w_{\boldsymbol{\pi}}$$

together with (4.1) to conclude that

$$(x_{n,\epsilon}^- \otimes \prod_{r=1}^{\ell} t^\epsilon (t^2 - a_r^2)^N) w_{\boldsymbol{\pi}^\sigma} = 0.$$

Hence

$$\left([x^-_{\theta_0}, x^-_{n,1}] \otimes \prod_{r=1}^{\ell} t(t^2-a_r^2)^N)\right) w^\sigma_{\boldsymbol{\pi}} = 0,$$

for some $N \gg 0$. Since the element $[x^-_{\theta_0}, x^-_{n,1}] \in \mathfrak{g}_1$ generates $\mathfrak{g}_1$ as a $\mathfrak{g}_0$–module and $[\mathfrak{n}^-, [x^-_{\theta_0}, x^-_{n,1}]] = 0$, we can now prove by similar arguments that for some $N \gg 0$,

$$(\mathfrak{g}_1 \otimes \prod_{r=1}^{\ell} t(t^2-a_r^2)^N) W(\boldsymbol{\pi}^\sigma) = 0.$$

Next, using the fact that $[x^-_{\theta_0}, \mathfrak{n}^-_1] = \mathbf{C}[x^-_{\theta_0}, x^-_{n,1}]$, we get

$$(x^-_{\theta_0} \otimes \prod_{r=1}^{\ell} (t^2-a_r^2)^N) W(\boldsymbol{\pi}^\sigma) = 0,$$

which finally gives

$$\left(\mathfrak{g}_0 \otimes \prod_{r=1}^{\ell} (t^2-a_r^2)^N \mathbf{C}[t^2, t^{-2}]\right) W(\boldsymbol{\pi}^\sigma) = 0,$$

and completes the proof.

□

Given positive integers $\ell, N \in \mathbf{Z}_+$, $\mathbf{a} = (a_1, \cdots, a_\ell) \in (\mathbf{C}^\times)^\ell$ and a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{a}) \subset \mathfrak{a}$, let

$$\mathfrak{a}^\sigma_{\mathbf{a},N} = L^\sigma(\mathfrak{g}) / \oplus_{\epsilon=0}^{m-1} (\mathfrak{a}_\epsilon \otimes t^\epsilon \prod_{k=1}^{\ell} (t^m - a_k)^N \mathbf{C}[t^m, t^{-m}]). \tag{4.3}$$

Corollary. *Let* $\boldsymbol{\pi}^\sigma = \prod_{r=1}^{\ell} \boldsymbol{\pi}^\sigma_{\lambda_r, a_r} \in \mathcal{P}^+_\sigma$ *be a standard decomposition of* $\boldsymbol{\pi}^\sigma$ *and set* $\mathbf{a} = (a_1, \cdots, a_\ell)$ *There exists* $N >> 0$ *such that*

$$W(\boldsymbol{\pi}^\sigma) = \mathbf{U}((\mathfrak{n}^-_{\mathbf{a}^m,N})^\sigma) w_{\boldsymbol{\pi}^\sigma}$$

4.2.

Proposition. *For all* $\boldsymbol{\pi}^\sigma \in \mathcal{P}^+_\sigma$, *the* $L^\sigma(\mathfrak{g})$*–module* $W(\boldsymbol{\pi}^\sigma)$ *is finite–dimensional.*

Proof. Let $u \in W(\boldsymbol{\pi}^\sigma)$ and write $u = y w_{\boldsymbol{\pi}^\sigma}$ for some $y \in \mathbf{U}(L^\sigma(\mathfrak{n}^-))$. The adjoint action of the subalgebras $\mathfrak{n}^\pm_0$ on $L^\sigma(\mathfrak{g})$ and hence on $\mathbf{U}(L^\sigma(\mathfrak{g}))$ is nilpotent. Using the defining relations we get immediately that for some $r = r(u) > 0$, we have

$$(x^\pm_\alpha \otimes 1)^r u = 0, \ \ \forall \ \ \alpha \in R^+_0.$$

This implies that $\mathbf{U}(\mathfrak{g}_0)u$ is a finite–dimensional $\mathfrak{g}_0$–submodule of $W(\boldsymbol{\pi}^\sigma)$, and hence $W(\boldsymbol{\pi}^\sigma)$ is isomorphic to a direct sum of $\mathfrak{g}_0$–modules. Write,

$$W(\boldsymbol{\pi}^\sigma) = \bigoplus_{\eta \in Q^+_0} W(\boldsymbol{\pi}^\sigma)_\mu,$$

where $W(\boldsymbol{\pi}^\sigma)_\mu = \{u \in W(\boldsymbol{\pi}^\sigma) : hu = \mu(h)u, \ \ \forall \ \ h \in \mathfrak{h}_0\}$. The representation theory of $\mathfrak{g}_0$ now implies that

$$W(\boldsymbol{\pi}^\sigma)_\mu \neq 0 \iff W(\boldsymbol{\pi}^\sigma)_{w(\mu)} \neq 0, \ \ \forall \ w \in W_0.$$

Since $W(\boldsymbol{\pi}^\sigma)_\mu = 0$ unless $\mu \in \lambda - Q_0^+$ and the number of elements in P_0^+ with this property is finite we get that $W(\boldsymbol{\pi}^\sigma)_\nu = 0$, for all but finitely many $\nu \in P_0^+$. The proposition follows if we prove that $\dim(W(\boldsymbol{\pi}^\sigma)_\nu) < \infty$ for all $\nu \in P_0^+$.

Choose $\mathbf{a}$ and N as in Corollary 4.1. Then

$$W(\boldsymbol{\pi}^\sigma)_\nu = \mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} w_{\boldsymbol{\pi}^\sigma}$$

where

$$\mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} = \{y \in \mathbf{U}((\mathfrak{n}^-_{\mathbf{a},N})^\sigma)_{\lambda_{\boldsymbol{\pi}}-\nu} : [h,y] = (\lambda_{\boldsymbol{\pi}} - \nu)(h)y, \ \ \forall \ h \in \mathfrak{h}_0\}.$$

Since this subspace is finite–dimensional it follows that $\dim(W(\boldsymbol{\pi}^\sigma)_\nu) < \infty$ as required. $\square$

4.3. Let $N \in \mathbf{Z}_+$ and $\mathbf{a} \in (\mathbf{C}^\times)^\ell$. The inclusion $\iota : L^\sigma(\mathfrak{g}) \to L(\mathfrak{g})$ obviously induces a Lie algebra map $\iota_{\mathbf{a},N} : \mathfrak{g}^\sigma_{\mathbf{a}^m,N} \to \mathfrak{g}_{\mathbf{a},N}$, where $\mathbf{a}^m = (a_1^m, \cdots, a_\ell^m)$. The following proposition will play a crucial role in the proof of Theorem 2.

Proposition. *Let $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ be such that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates. For all $N \in \mathbf{Z}_+$ we have an isomorphism of Lie algebras,*

$$\mathfrak{g}_{\mathbf{a},N} \cong \bigoplus \mathfrak{g}^\sigma_{a_i^m,N} \cong \mathfrak{g}^\sigma_{\mathbf{a}^m,N}$$

for all $N \in \mathbf{Z}_+$. In particular, the composite map $L^\sigma(\mathfrak{g}) \to L(\mathfrak{g}) \to \mathfrak{g}_{\mathbf{a},N}$ is surjective.

Proof. The proof that

$$\bigoplus \mathfrak{g}^\sigma_{a_i^m,N} \cong \mathfrak{g}^\sigma_{\mathbf{a}^m,N}$$

is an obvious modification of the one given in Lemma 2.1 which also shows now that it is sufficient to prove the proposition when $\ell = 1$. For this, let $a \in \mathbf{C}^\times$ and $f = t^\epsilon g$ where $g \in \mathbf{C}[t^m, t^{-m}]$. Then,

$$f \in (t-a)^N \mathbf{C}[t, t^{-1}] \quad \iff \quad f \in t^\epsilon (t^m - a^m)^N \mathbf{C}[t^m, t^{-m}],$$

which proves that $\iota_{a,N}$ is injective. The proposition follows by noting that

$$\dim \mathfrak{g}^\sigma_{a^m,N} = \dim \mathfrak{g}_{a,N} = N \dim \mathfrak{g}.$$

$\square$

4.4. We note some elementary observations which we use without further comment. Any $\mathfrak{g}_{\mathbf{a},N}$–module (resp. $\mathfrak{g}^\sigma_{\mathbf{a},N}$) is obviously a $L(\mathfrak{g})$–module (resp. $L^\sigma(\mathfrak{g})$–module). Moreover if $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ is such that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates then for all $N \in \mathbf{Z}_+$, any $\mathfrak{g}_{\mathbf{a},N}$–module V is also a $\mathfrak{g}_{\mathbf{a}^m,N}$–module and we write it as $V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}}$. Similarly if we start with a $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module V we get a $\mathfrak{g}_{\mathbf{a},N}$–module which we write as $V_{\mathfrak{g}_{\mathbf{a},N}}$. Note also that if V is an $\mathfrak{g}_{\mathbf{a},N}$–module, then

$$(V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}})_{\mathfrak{g}_{\mathbf{a},N}} \cong_{\mathfrak{g}_{\mathbf{a},N}} V, \qquad (V_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}})_{\mathfrak{g}_{\mathbf{a},N}} \cong_{L(\mathfrak{g})} V. \tag{4.4}$$

4.5.

Lemma. *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$, and assume that $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$.*

(i) *There exists $\ell, N \in \mathbf{Z}_+$ and $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ with $\mathbf{a}$ and $\mathbf{a}^m$ having distinct coordinates such that $W(\boldsymbol{\pi})$ and $W(\boldsymbol{\pi}^\sigma)$ are modules for both $\mathfrak{g}_{\mathbf{a},N}$ and $\mathfrak{g}_{\mathbf{a}^m,N}$.*

(ii) *In particular,*

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}},$$

and $V(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}}$ is an irreducible $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module.

Proof. Let $\boldsymbol{\pi} = \prod_{k=1}^\ell \boldsymbol{\pi}_{\lambda_k,a_k}$, where $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{a}^m$ have distinct coordinates. Proposition 2.7 implies that $W(\boldsymbol{\pi}) = \mathbf{U}(\mathfrak{g}_{\mathbf{a},N})w_{\boldsymbol{\pi}}$. Using proposition 4.3 we see that $W(\boldsymbol{\pi})$ is also module for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$ and so we get

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}}.$$

Similarly Proposition 4.1 implies that $W(\boldsymbol{\pi}^\sigma)$ is a module for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$ and hence for $\mathfrak{g}_{\mathbf{a},N}$. Since $V(\boldsymbol{\pi})$ is an irreducible module for $\mathfrak{g}_{\mathbf{a},N}$, it follows that it is also irreducible as a $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–module and the proposition is proved. □

The following proposition proves (i) of Theorem 2.

Proposition. *Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma^+$, $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$.*

(i) *Regarded as $L^\sigma(\mathfrak{g})$–module $W(\boldsymbol{\pi})$ is a quotient of $W(\boldsymbol{\pi}^\sigma)$ and hence*

$$V(\boldsymbol{\pi}) \cong_{L^\sigma(\mathfrak{g})} V(\boldsymbol{\pi}^\sigma).$$

(ii) *For $N \gg 0$, the $L^\sigma(\mathfrak{g})$–module structure on $W(\boldsymbol{\pi}^\sigma)$ (resp. $V(\boldsymbol{\pi}^\sigma)$) extends to an $L(\mathfrak{g})$–module action on $W(\boldsymbol{\pi}^\sigma)$ (resp. $V(\boldsymbol{\pi}^\sigma)$).*

(iii) *The module $W(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}$ is a $L(\mathfrak{g})$–module quotient of $W(\boldsymbol{\pi})$.*

Proof. Using (3.1), (3.2) and the fact that $\mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma$, we see that $w_{\boldsymbol{\pi}}$ satisfies the defining relations of $W(\boldsymbol{\pi}^\sigma)$. Part (i) follows if we prove that $W(\boldsymbol{\pi}) = \mathbf{U}(L^\sigma(\mathfrak{g}))w_{\boldsymbol{\pi}}$. But this is true because proposition 4.3 and proposition 4.5 prove that there exists $\mathbf{a} \in (\mathbf{C}^\times)^\ell$ such that

$$W(\boldsymbol{\pi})_{\mathfrak{g}^\sigma_{\mathbf{a}^m,N}} = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})w_{\boldsymbol{\pi}} = \mathbf{U}(L^\sigma(\mathfrak{g}))w_{\boldsymbol{\pi}}.$$

It now follows that $V(\boldsymbol{\pi})_{\mathfrak{g}_{\mathbf{a}^m,N}}$ is the irreducible quotient of $W(\boldsymbol{\pi}^\sigma)$ and hence is isomorphic to $V(\boldsymbol{\pi}^\sigma)$ as $L^\sigma(\mathfrak{g})$–modules.

To prove (ii), note that that we have a surjective homomorphism of Lie algebras

$$\mathbf{p} : L(\mathfrak{g}) \to \mathfrak{g}_{\mathbf{a},N} \to \mathfrak{g}^\sigma_{\mathbf{a}^m,N},$$

such that the restriction of $\mathbf{p}$ to $L^\sigma(\mathfrak{g})$ is just the canonical surjection. Moreover

$$\mathbf{p}(L(\mathfrak{n}^\pm)) \subset (\mathfrak{n}^\pm)^\sigma_{\mathbf{a}^m,N}, \quad \mathbf{p}(L(\mathfrak{h})) \subset \mathfrak{h}^\sigma_{\mathbf{a}^m,N},$$

and hence

$$L(\mathfrak{n}^+)w_{\boldsymbol{\pi}^\sigma} = 0, \quad L(\mathfrak{h})w_{\boldsymbol{\pi}^\sigma} = \mathbf{C}w_{\boldsymbol{\pi}^\sigma}.$$

Since $\dim(W(\boldsymbol{\pi}^\sigma)) < \infty$, it follows from Theorem 1(i) that $W(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}$ is a quotient of $W(\tilde{\boldsymbol{\pi}})$ for some $\tilde{\boldsymbol{\pi}} \in \mathcal{P}^+$. Since the module $W(\tilde{\boldsymbol{\pi}})$ has an unique irreducible quotient $V(\tilde{\boldsymbol{\pi}})$, part (iii) follows if we prove that

$$V(\boldsymbol{\pi}) \cong_{L(\mathfrak{g})} V(\boldsymbol{\pi}^\sigma)_{\mathfrak{g}_{\mathbf{a},N}}.$$

But this follows from part (i) and (4.4) and part (iii) is now proved. □

4.6. The next proposition proves part (ii) of Theorem 2.

Proposition. *Let $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k} \in \mathcal{P}^+_\sigma$ and assume that $\mathbf{a}$ and $\mathbf{a}^m$ have distinct coordinates. As $L^\sigma(\mathfrak{g})$–modules, we have*

$$W(\boldsymbol{\pi}^\sigma) \cong \bigotimes_{k=1}^{\ell} W\left(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}\right).$$

Proof. For $1 \le k \le \ell$, set

$$\boldsymbol{\pi}^\sigma_k = \prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}.$$

It is checked easily that the element $\otimes_{k=1}^{\ell} w_{\boldsymbol{\pi}^\sigma_k}$ satisfies the defining relations of $w_{\boldsymbol{\pi}^\sigma}$ and hence we have an $L^\sigma(\mathfrak{g})$–module map,

$$\eta : W(\boldsymbol{\pi}^\sigma) \to \bigotimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k).$$

The proposition follows if we prove that this map is surjective. For then, taking $\boldsymbol{\pi} \in \mathbf{i}(\boldsymbol{\pi}^\sigma)$ and $\boldsymbol{\pi}_k \in \mathbf{i}(\boldsymbol{\pi}^\sigma_k)$, we have

$$\dim W(\boldsymbol{\pi}) = \dim W(\boldsymbol{\pi}^\sigma) \ge \prod_{k=1}^{\ell}\dim W(\boldsymbol{\pi}^\sigma_k) = \prod_{k=1}^{\ell}\dim W(\boldsymbol{\pi}_k) = \dim W(\boldsymbol{\pi}),$$

where the first equality uses part (i) of Theorem 2 and the last equality follows from Theorem 1(iii). To prove that η is surjective choose $N >> 0$ so that $W(\boldsymbol{\pi}^\sigma)$ is a module for $\mathfrak{g}_{\mathbf{a}^m,N}$ and also so that for all $1 \le k \le \ell$ the algebra $\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N}$ acts on $W(\boldsymbol{\pi}^\sigma_k)$ where $\mathbf{a}^m_k = \{a^m_k\}$ and we have

$$W(\boldsymbol{\pi}^\sigma_k) = \mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N})w_{\boldsymbol{\pi}^\sigma_k}.$$

On the other hand by Proposition 4.3 we have

$$\mathfrak{g}^\sigma_{\mathbf{a}^m,N} \cong \bigoplus_{k=1}^{\ell}\mathfrak{g}^\sigma_{\mathbf{a}^m_k,N},$$

and hence $\otimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k)$ is cyclic for $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$, i.e

$$\mathbf{U}(\mathfrak{g}^\sigma_{\mathbf{a}^m,N})(\otimes_{k=1}^{\ell} w_{\boldsymbol{\pi}^\sigma_k}) = \bigotimes_{k=1}^{\ell} W(\boldsymbol{\pi}^\sigma_k).$$

This proves that η is a surjective map of $\mathfrak{g}^\sigma_{\mathbf{a}^m,N}$–modules and the proof of the proposition is complete.

□

4.7. We now prove (iii) of Theorem 2. Recall that in Section 2.1, we have identified elements of P_σ^+ with elements of P^+ and hence for each $a \in \mathbf{C}^\times$ and $\lambda \in P_\sigma^+$ we have elements $\boldsymbol{\pi}_{\lambda,a} \in \mathcal{P}^+$ and $\boldsymbol{\pi}^\sigma_{\lambda,a} \in \mathcal{P}_\sigma^+$. Moreover, $\boldsymbol{\pi}_{\lambda,a} \in \mathbf{i}(\boldsymbol{\pi}^\sigma_{\lambda,a})$.

Proof. Choose $b_\epsilon \in \mathbf{C}^\times$, $0 \le \epsilon \le m-1$ such that

$$b_r \neq b_s, \quad b_r^m \neq b_s^m, \quad r \neq s.$$

Using Lemma 3.7, Theorem 2(ii) and Theorem 2(i) in that order gives,

$$\bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \cong_{\mathfrak{g}_0} \bigotimes_{\epsilon=0}^{m-1} W(\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}) \cong_{L^\sigma(\mathfrak{g})} W(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}).$$

Since $\lambda_\epsilon \in \mathcal{P}_\sigma^+$, we have $\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon} \in \mathbf{i}(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon})$ and so by Theorem 2(i) we get,

$$W(\prod \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,b_\epsilon}) \cong_{L^\sigma(\mathfrak{g})} W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}).$$

Theorem 1 gives,

$$W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}) \cong_{\mathfrak{g}} W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),1}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{\sum_{\epsilon=0}^{m-1} \sigma^\epsilon(\lambda_\epsilon),a}).$$

And since $\boldsymbol{\pi}_{\sum_{\epsilon=0}^{m-1} \sigma^\epsilon(\lambda_\epsilon),a} \in \mathbf{i}(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a})$, we get

$$W(\prod \boldsymbol{\pi}_{\sigma^\epsilon(\lambda_\epsilon),\zeta^\epsilon b_\epsilon}) \cong_{\mathfrak{g}_0} W(\boldsymbol{\pi}_{\lambda,a}) \cong_{L^\sigma(\mathfrak{g})} W(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}),$$

which completes the proof.

□

4.8. We now prove Theorem 2 (iv). By Theorem 2(i), we have

$$W^\sigma(\boldsymbol{\pi}^\sigma_{\lambda,a}) \cong_{L^\sigma(\mathfrak{g})} W(\boldsymbol{\pi}_{\lambda,a}).$$

Theorem 1 gives if $\mathfrak{g}$ not of type A_{2n} that

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} \bigotimes_{i=1}^{n} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i}$$

and for $\mathfrak{g}$ of type gives A_{2n}

$$W(\boldsymbol{\pi}_{\lambda,a}) \cong_{\mathfrak{g}} W(\boldsymbol{\pi}_{2\omega_n,1})^{\otimes \frac{m_n}{2}} \otimes \bigotimes_{i=1}^{n-1} W(\boldsymbol{\pi}_{\omega_i,1})^{\otimes m_i},$$

which completes the proof.

4.9. We now prove Theorem 2(v). This part of the proof is very similar to the one given in [6] in the untwisted case and we shall only give a sketch of the proof. Thus, let V be an $L^\sigma(\mathfrak{g})$–module, assume that V is finite–dimensional and that it is generated by an element $v \in V$ such that

$$L^\sigma(\mathfrak{n}^+)v = 0, \quad \mathbf{U}(L^\sigma(\mathfrak{h}))v = \mathbf{C}v.$$

Let $\lambda \in P_\sigma^+$ be such that $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$. Since V is finite–dimensional it follows from the representation theory of the subalgebras $\{x_{i,0}^\pm, h_{i,0}\}$, $i \in I_0$ that $\lambda \in P_\sigma^+$ and also that

$$(x_{i,0}^-)^s = 0, \;\; i \in I_0, \quad s \in \mathbf{Z}_+, \quad s \geq \lambda(h_i) + 1 \tag{4.5}$$

Moreover if $\mathfrak{g}$ is of type A_{2n}, we find by working with the subalgebra $\left\{\frac{1}{2}h_{n,0}, y_{n,1}^\pm \otimes t^{\mp 1}\right\}$ that

$$(y_{n,1}^- \otimes t)^s v = 0, \quad s \in \mathbf{Z}_+, \quad s \geq \frac{1}{2}\lambda(h_{n,0}) + 1. \tag{4.6}$$

Applying $(x_{i,0}^+ \otimes t)^s$ to both sides of (4.5), ($i \neq n$ if $\mathfrak{g}$ of type A_{2n}) we find by using Lemma 3.3 (i), (ii), that

$$\left(\mathbf{p}_{i,\sigma}^+(u)\right)_s = 0, \quad s > \lambda(h_{i,0}),$$

while if $\mathfrak{g}$ is of type A_{2n}, we apply $(x_{n,0}^+)^{2s}$ to both sides of (4.6) and using Lemma 3.3(iii), we find

$$\left(\mathbf{p}_{n,\sigma}^+(u)\right)_k = 0, \quad k > \lambda(\frac{1}{2}h_{n,0}).$$

Set

$$\pi_i^\sigma(u) = \sum_{k=0}^\infty (\mathbf{p}_i^\sigma(u))_k u^k,$$

and let $\boldsymbol{\pi}^\sigma = (\pi_i^\sigma)_{i \in I_0}$. The preceding arguments show that $\boldsymbol{\pi}^\sigma$ is an I_0–tuple of polynomials. We claim that

$$\lambda = \lambda_{\boldsymbol{\pi}}, \quad \mathbf{p}_{i,\sigma}^-(u)v = (\pi_i^\sigma(u))^- v, \tag{4.7}$$

which now shows that V is a quotient of $W(\boldsymbol{\pi}^\sigma)$. To prove that $\lambda = \lambda_{\boldsymbol{\pi}}$ is equivalent to proving that

$$\left(\mathbf{p}_{i,\sigma}^+\right)_{\lambda(h_i)} v \neq 0, \tag{4.8}$$

for all $i \in I$, if if $\mathfrak{g}$ is not of type A_{2n} and for all $i \neq n$ if $\mathfrak{g}$ is of type A_{2n} and if $\mathfrak{g}$ is of type A_{2n}

$$\left(\mathbf{p}_{n,\sigma}^+(u)\right)_{\frac{1}{2}\lambda(h_{n,0})} .v \neq 0. \tag{4.9}$$

It is now easy to see (keeping in mind that $\left(\mathbf{p}_{i\sigma}(u)\right)_0 = 1$) that the following Lemma implies (4.7).

Lemma. *Let V be a finite–dimensional $L^\sigma(\mathfrak{g})$–module and let $v \in V_\lambda)$ be such that $L^\sigma(\mathfrak{n}^+)v = 0$. For all $i \in I_0$ ($i \neq n$ for $\mathfrak{g}$ of type A_{2n}), we have*

$$(\mathbf{p}_{i,\sigma}^+(u))_{\lambda(h_{i,0})}(\mathbf{p}_{i,\sigma}^-(u))_k.v = (\mathbf{p}_{i,\sigma}^+(u))_{\lambda(h_{i,0})-k}.v, \quad 0 \leq k \leq \lambda(h_{i,0}),$$

and for $\mathfrak{g}$ of type A_{2n}, we have

$$(\mathbf{p}_{n,\sigma}^+(u))_{\frac{1}{2}\lambda(h_{n,0})}(\mathbf{p}_{n,\sigma}^-(u))_k.v = (\mathbf{p}_{n,\sigma}^+(u))_{\frac{1}{2}\lambda(h_{n,0})-k}.v, \quad 0 \leq k \leq \frac{1}{2}\lambda(h_{n,0}).$$

Proof. The proof of the first statement is given in [6, Proposition 1.1] and the key ingredient in that proof is Lemma 3.4 (i). The proof when $i = n$ and $\mathfrak{g}$ of type A_{2n} is entirely similar and one uses Lemma 3.4 (iii)(a) with $r = \frac{1}{2}\lambda(h_{n,0}) + 1$. □

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